COMPUTATIONAL METHODS FOR COMPRESSIBLE FLOW PROBLEMS

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Summary

In this chapter, we provide an introduction to the numerical schemes devoted to the simulation of compressible flow problems. We first review a few elements of modeling of compressible flow problems and then provide a brief description of the solutions to one dimensional problems. We identify a class of partial differential equations (PDEs) which share common features with those describing flow problems. In particular, we show that it is not possible to expect globally smooth solutions. Then we describe a class of numerical schemes that are able to approximate the solutions, first for one-dimensional problems and for the multidimensional problems. Finally, numerical examples are provided.

1. Introduction

In many industrial applications, fluid flow cannot be approximated by the incompressible Navier Stokes equations: either the speed is too high (e.g. flows around aircraft or reentry vehicles), or the density is not uniform (e.g. in car engines), or acoustic effects are not negligible (e.g. when an aircraft is landing). In these cases and many others, one has to rely on the Navier Stokes equations for compressible flow., For simplicity, in this chapter we focus on flows where viscous effects can be neglected.

The state of flow is represented by a vector of conserved variables $W = (\rho, \rho\vec{u}, E)^T$ where ρ is the density, $\vec{u}$ is the flow velocity, and $E = \rho\varepsilon + \frac{1}{2}\rho\vec{u}^2$ is the total energy. Here, ε represents the specific internal energy. in the light of the basic principles of continuum mechanics, W satisfies the following:

$$\frac{\partial q}{\partial t} + \operatorname{div} \vec{F}(q) = 0 \qquad \text{for} \quad x \in \Omega \qquad \text{and} \quad t > 0$$

$$q(x,0) = q_0(x) \qquad\qquad \text{for} \quad x \in \Omega \tag{1}$$

In (1), Ω is an open subset of $\mathbb{R}^N$, $N = 1,2,3$, the flux $\vec{F}$ is given by

$$\vec{F}(q) = \begin{pmatrix} \rho\vec{u} \\ \rho\vec{u}\otimes\vec{u} + p\,\mathrm{Id} \\ (E+p)\vec{u} \end{pmatrix} \tag{2}$$

The system is closed by a relation between the pressure p and the state variables. In this paper, we consider the simplest example of an ideal gas for which

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho\vec{u}^2\right). \tag{3}$$

In (3), γ is the ratio of specific heats and is γ =1.4 in all applications we present in the paper

The first relation given by (1), namely,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{u}) = 0 \tag{4}$$

signifies conservation of mass. The second relation,

$$\frac{\partial(\rho\vec{u})}{\partial t} + \operatorname{div}(\rho\vec{u}\otimes\vec{u}) + \operatorname{div} \quad p = 0,$$

is for conservation of momentum and application of the fundamental principles of mechanics. The nest relation is for conservation of total energy,

$$\frac{\partial E}{\partial t} + \operatorname{div}((E+p)\vec{u}) = 0,$$

and represents the first law of thermodynamics.

There is yet another relation, namely the second law of thermodynamics. It states that the entropy of a fluid particle increases with time, that is

$$\frac{\partial s}{\partial t} + \vec{u}\,.\,\nabla s \geq 0. \tag{5}$$

By multiplying (5) by ρ and (4) by s, and then summing up, we get

$$\frac{\partial \rho s}{\partial t} + \mathrm{div}(\rho s \vec{u}) \geq 0. \tag{6}$$

In (5) and (6), s represents specific entropy. In the case of an ideal gas, we have

$$s = s_0 + c_\upsilon \log(p\rho^{-\gamma}). \tag{7}$$

One of its important properties is that the function $s = s(\rho, E - \frac{1}{2}\rho\vec{u}^2)$ is a concave function of $(\rho, E - \frac{1}{2}\rho\vec{u}^2)$. The problem of relation (5) is that it is written in non-conservative form. In particular, as we see later, it is natural to admit solution for which neither $\vec{u}$ nor s are continuous, so that the product $\vec{u}\,.\,\nabla s$ may be undefined in general. Interest in relation (7) is two fold. First, it is written in conservative form so that, as we see later, the problem of non-smooth solution can be solved. Second, the mathematical entropy, $S = -\rho s(\rho, \vec{u}, E)$ is a convex function of its arguments.

2. A Brief Description of the Solutions

It is very natural to consider non continuous solutions. Take for example the situation of a shock tube. This is an experimental device which is used to generate shock waves. It is a tube filled with gas. It is initially made of two sections separated by a diaphragm, see Figure 1. The left and right chambers are filled with gas with different velocity and pressure. In this experimental setup, the initial velocity is set to zero; the 1D-Riemann problem,

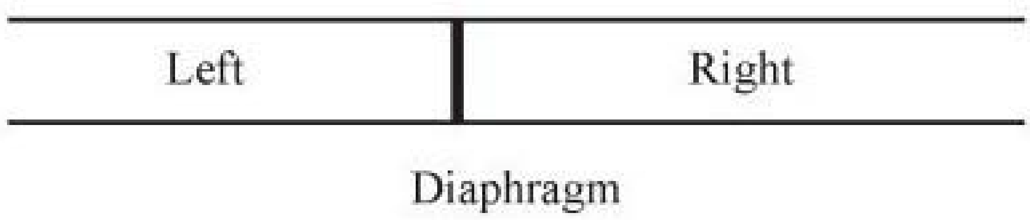

Figure 1: the shock tube problem

$$\begin{aligned} &\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad\qquad x \in \mathbb{R}, t > 0 \\ &q(x,0) = \begin{cases} q_L & \text{if } x < 0 \\ q_R & \text{if } x > 0 \end{cases} \end{aligned} \tag{8}$$

plays a very important role in the numerical simulation of compressible flows. In the Riemann problem, the initial condition is discontinuous.

An analysis of the solution of the shock tube problem shows that it is equivalent to the superposition of two piston problems, see Figure 2.

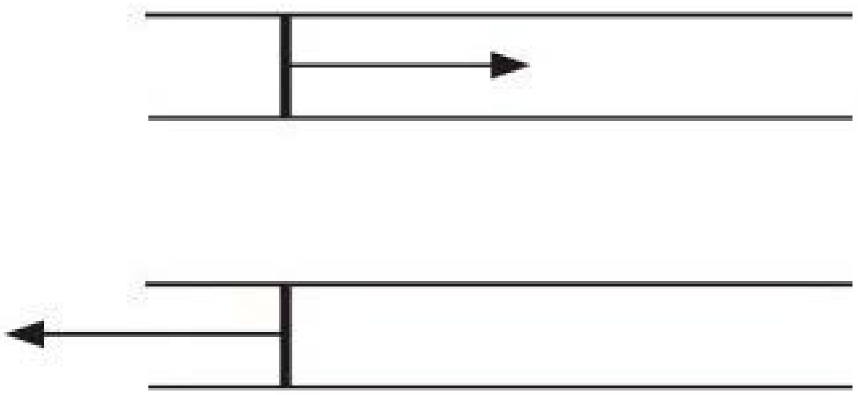

Figure 2: The piston problems

In one case, the piston is moving to the left, in the other case, it is moving to the right. If one considers an experiment with a tube filled with gas where a piston, at time $t = 0$ moves with a velocity V to the left, then the density and the pressure in the vicinity of the piston will decrease, and a wave is created from the piston. If we sit far enough from the piston, we do not feel the effect of changing the pressure: in general, this wave connects the area where the solution is still uniform and the fan. The connection is continuous but not continuously differentiable.

If one now pushes the piston to the right, one can show that the state flow parameters are uniform in the neighborhood of the piston, also uniform far enough, and the two uniform states are connected by a shock wave, that is a moving discontinuity where all the flow parameters are discontinuous.

If the piston is moved smoothly, depending on the conditions, the flow may be continuous at any time or discontinuities can be generated.

In the case where the solution is not smooth, what is the meaning of the system (1) ? The answer has been provided by P.Lax.

2.1 On the Notion of Solutions

The idea is to integrate (1) against compactly supported smooth test functions $\varphi \in C_c^1(\Omega \times [0,T])^{N+2}$. Assuming that q is smooth, if we integrate (1) multiplied by φ, we get

$$\int_{\Omega\times[0,T]}\left(q\cdot\frac{\partial\varphi}{\partial t}+\vec{F}(q)\cdot\nabla\varphi\right)dxdt-\int_{\Omega}q(x,0)\ .\ \varphi(x,0)\,dx=0 \qquad (9)$$

The advantage of (9) w.r.t. (1) is that q does not appear anymore with derivatives: they have been transferred to the terms containing φ. Thanks to this, we can define the *weak solutions* of (1) as the functions in $L^{\infty}(\Omega\times[0,T])\cap L^{1}(\Omega\times[0,T])$ such that for any $\varphi\in C_c^1(\Omega\times[0,T])^{N+2}$, (9) holds.
Similarly, the inequality (7) has to be understood as the solutions q such that for any $\varphi\in C_c^1(\Omega\times[0,T]),\varphi\geq 0$, one has

$$\int_{\Omega\times[0,T]}\left(\rho s\frac{\partial\varphi}{\partial t}+\rho s\vec{u}\ .\ \nabla\varphi\right)dxdt-\int_{\Omega}s(x,0)\,\varphi(x,0)\,dx\geq 0. \qquad (10)$$

This relation shows the advantage of (5) with respect to (6). Solutions that satisfies the inequalities (6) for all the convex entropy U associated to an entropy flux G, that is

$$\nabla\vec{G}=\nabla_q U\ .\ \nabla_W\vec{F}$$

and

$$\int_{\Omega\times[0,T]}\left(U\frac{\partial\varphi}{\partial t}+\vec{G}\ .\ \nabla\varphi\right)dxdt-\int_{\Omega}U(x,0)\,\varphi(x,0)\,dx\geq 0.$$

are called entropy solutions. For a characterization of the couple entropy-flux for the Euler equations, (U,G), see [15].

The solutions we are interested in are piecewise smooth. The discontinuities of the solutions lie on a piecewise smooth curve $\sum$, see Figure 3. By using test functions whose supports are localized in the neighborhood of $\sum$, we get the Rankine-Hugoniot relations

$$[F_{\vec{n}}(q)]=\sigma[q] \qquad (11)$$

where $[f]=(.)^{+}-(.)^{-}$ is the jump of f at $M\in\sum,F_{\vec{n}}$ is the normal flux and σ is the speed of the discontinuity. The inequality (7) becomes

$$[\rho\vec{u}\ .\vec{n}s]\geq\sigma[s] \qquad (12)$$

In (12), the definition of $[f]$ is important because of the inequality. Equation (12) indicates that the specific entropy of a fluid particle increases across a discontinuity.

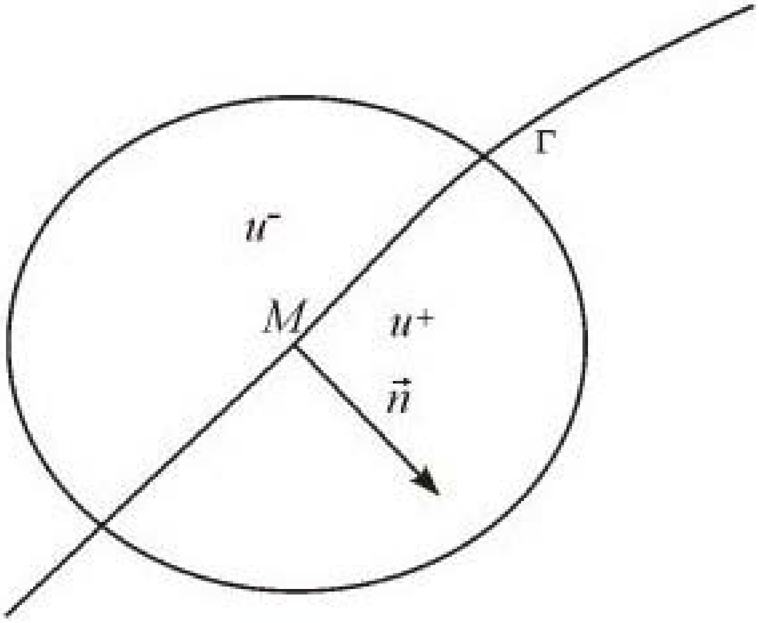

Figure 3: Discontinuous solutions on a piecewise smooth curve

Tadmor [32] showed that the solution (if it is bounded) satisfies the following minimum principle

$$s(x,t) \geq \min_{\|y-x\| \leq t \|\vec{u}\|_\infty} s(y,0) \tag{13}$$

where $\|x\|$ is the Euclidean norm of x and $\|\vec{u}\|_\infty$ is the L^∞ norm of the velocity field

2.2 Hyperbolicity: Characteristic Form of the Equations

A straightforward calculation shows that if $\vec{n}(n_x, n_y)$ is any vector, then the matrix $n_x A + n_y B$, where A and B are the Jacobian matrices of the flux, is diagonalizable in $\mathbb{R}$. The system (1) is said to be hyperbolic. More precisely, set c the speed of sound

$$c := \left.\frac{\partial p}{\partial \rho}\right|_s = \sqrt{\gamma \frac{p}{\rho}}.$$

The eigenvalues of $n_x A + n_y B$ are $\lambda_0 = \vec{u}.\vec{n}, \lambda_\pm = \vec{u}.\vec{n} \pm c$. The eigenvalue λ_0 is double (triple in 3D). If $\vec{t} = (-ny, n_x)$, the eigenvectors are respectively

$$r_0^n = \begin{pmatrix} 1 \\ u \\ \upsilon \\ \dfrac{u^2+\upsilon^2}{2} \end{pmatrix} \quad r_0^t = \begin{pmatrix} 0 \\ -n_y \\ n_x \\ \vec{n}.\vec{t} \end{pmatrix} \quad r_\pm \begin{pmatrix} 1 \\ u \pm n_x c \\ \upsilon \pm n_y c \\ H \pm \vec{u}.\vec{n} \end{pmatrix},$$

we skip the left eigenvectors $\ell_0^n, \ell_0^t, \ell_\pm$, they can be found in [13].

It is possible to find special solutions to (1) of the form $q = q(\sigma), \sigma \in \mathbb{R}$ such that

$$\frac{dq}{d\sigma} = r(q(\sigma))$$

where r is any of the eigenvectors of $n_x A + n_y B$ and $\sigma = \sigma(x\ .\ \vec{\omega}, t)$, $\vec{\omega}$ being a fixed direction. We are looking for planar waves. Using (1), we see that $(\chi, t) \mapsto \sigma(\chi, t)$ satisfies an equation of the type

$$\frac{\partial \sigma}{\partial t} + \lambda(q(\sigma, \vec{\omega}, t)) \frac{\partial \sigma}{\partial \chi} = 0\,.$$

These three equations are non linear scalar equations. These particular solutions are characterized as follows

1. waves associated with $\lambda = u \pm c$, these are fans where $p\rho^{-\gamma}$ and $u \mp \dfrac{2}{\gamma - 1}c$ remain constant
2. waves associated with $\lambda = u, u$ and p remain constant.

The functions $p\rho^{-\gamma}, u \mp \dfrac{2}{\gamma-1}c, u$ and p are the Riemann invariants.

When analyzing the structure of the Rankine Hugoniot relations, it can be easily seen that the piecewise constant solutions must be of the form

1 either u and p remain constant, ρ and s can be arbitrary: these are contact discontinuities;

2 or all the variables are discontinuous, they are shock waves. Let us denote with a superscript + the post-shock conditions and with a superscript – the pre-shock conditions and denoting by σ is the shock speed and $\upsilon = u - \sigma$, and $M = \dfrac{\upsilon^-}{c^-}$ the relative match number of the shock, we have

$$\frac{\rho^-}{\rho^+} = \frac{2}{\gamma+1}\frac{1}{M^2} + \frac{\gamma-1}{\gamma+1}$$

$$\frac{\upsilon^+}{\upsilon^-} = \frac{2}{\gamma+1}\frac{1}{M^2} + \frac{\gamma-1}{\gamma-1}$$

$$\frac{p^+}{p^-} = \frac{2\gamma}{\gamma+1}M^2 - \frac{\gamma-1}{\gamma+1}$$

From this we see that the density and the pressure increase through a shock. The opposite is true for the velocity.

We observe that u and p are invariant parameters for the wave associated with $\lambda = u$. This remark is useful in the effective solution of the Riemann problem.

These particular solutions are building blocks of the solution of the Riemann problem 8. They can be observed in Figures 7, 8, and 9.

More on the structure of the solution can be found in [13, 28].

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